

# Imaginary Entanglement as Cost of Unitarity

Constantin V. Usenko<sup>1,\*</sup>

<sup>1</sup>*National Shevchenko University of Kyiv, Theoretical Physics Department*

This report is about contradiction between fidelity needed to determine the entanglement and concomitant noise that always accompanies precise measurement.

Account of quantum properties of field leads to additional noise caused by multiple particle creation through nonunitarity of quantum field representation in embedded sections of space (Unruh noise).

Causes of quantum noise vanish at leaving off assumption about statistical independence of detectors. Smearing of detector leads to elimination of causes of Unruh noise and to emergence of imaginary entanglement of few mode states caused by overlap of detector sections.

## 1. INTRODUCTION

Progress in creation of essentially quantum states of electromagnetic field (entangled states of few photons) leads to need in analysis of physical meaning of the basic concept of quantum physics - concept of probability distribution of particle registration [1, 2] by several placed near each other detectors.

Well known mathematical scheme of determination of probability distribution deals with probabilities of detection of particles in the sequence of embedded sections (Rokhlin scheme) in implicit assumption that detectors are statistically independent. One expects that at decrease of the size of detector count decreases proportionally. The constant

of proportionality is taken as probability density (probability per size).

Account of quantum properties of field leads to additional noise [9] caused by multiple particle creation [3, 4] through nonunitarity of quantum field representation in embedded sections of space [5] (Unruh noise).

Causes of quantum noise vanish at leaving off assumption about statistical independence of detectors. Smearing of detector leads to elimination of causes of Unruh noise and to emergence of imaginary entanglement of few mode states caused by overlap of detector sections.

This report is about contradiction between fidelity needed to determine the entanglement and concomitant noise that always accompanies precise measurement.

---

\* usenko@univ.kiev.ua

Specific objects in the study of entanglement of states are pairs of polarized photons (bi-photons especially) and electrons with opposite spins.

Typical procedure of study of entanglement is in consideration of entangled electron pair moving to pair of spin state detectors. There exists some probability for the state "UP-Down" and some probability for the opposite state.

Identity of particles is one more reason for entanglement. The process in which the upper particle is registered by down detector and vice versa is identical to process in which each particle falls to its own detector.

## 2. ENTANGLEMENT IN MEASUREMENT FOR TWO PARTS

Mathematical model is based on the study of the density matrix properties.

Mathematical model of the measuring unit is expansion of unity forming positive operator-valued measure in the space of states of the system. The device measuring the states of the first sub-system does not detect the state difference of the second one, and vice versa.

$$\begin{aligned} M_A &= M^{(a)} \otimes 1^{(b)} & M_B &= 1^{(a)} \otimes M^{(b)} \\ M^{(a)} \left[ \hat{\rho}_1^{(a)} \right] &= M_{a1} & M^{(b)} \left[ \hat{\rho}_1^{(b)} \right] &= M_{b1} \\ M^{(a)} \left[ \hat{\rho}_2^{(a)} \right] &= M_{a2} & M^{(b)} \left[ \hat{\rho}_2^{(b)} \right] &= M_{b2} \end{aligned}$$

### 2.1. Independent parts

The system consisting of independent parts has as density matrix direct product of density matrices of separate parts

$$\hat{\rho}^{(a+b)} = \left( p_a \hat{\rho}_1^{(a)} + (1-p_a) \hat{\rho}_2^{(a)} \right) \otimes \left( p_b \hat{\rho}_1^{(b)} + (1-p_b) \hat{\rho}_2^{(b)} \right).$$

As the result, reaction of the device on the system consisting of independent parts is equal to the reaction on its own sub-system. Probability of registration of some state of the system is equal to the product of probabilities for respective states of the sub-systems  $P(a1) = p_a$ ,  $P(a2) = (1-p_a)$ ,  $P(b1) = p_b$ ,  $P(b2) = (1-p_b)$ .

$$\begin{aligned} P(a1 \&\& b1) &= p_a p_b; \\ P(a2 \&\& b1) &= (1-p_a) p_b; \\ P(a1 \&\& b2) &= p_a (1-p_b); \\ P(a2 \&\& b2) &= (1-p_a) (1-p_b); \\ P(a_k \&\& b_m) &= P(a_k) P(b_m). \end{aligned}$$

### 2.2. Entangled state

Entangled states are characterized by density matrix being unreduceable mixture of the products of sub-system states

$$\hat{\rho}^{(a+b)} = p \hat{\rho}_{1a}^{(a)} \otimes \hat{\rho}_{1b}^{(b)} + (1-p) \hat{\rho}_{2a}^{(a)} \otimes \hat{\rho}_{2b}^{(b)} \quad (1)$$

Probability of registration of some state of one sub-system essentially depends on the result of measurement for another one, and joint probability distribution is different from

product of probability distributions for each of the sub-systems

$$\begin{aligned} P(a1 \& b1) &= p \neq P(a1)P(b1); \\ P(a2 \& b1) &= 0 \neq P(a2)P(b1); \\ P(a1 \& b2) &= 0 \neq P(a1)P(b2); \\ P(a2 \& b2) &= (1-p) \neq P(a2)P(b2), \end{aligned}$$

and mutual probabilities are

$$\begin{aligned} P(a1|b1) &= 1; P(a2|b1) = 0; \\ P(a1|b2) &= 0; P(a2|b2) = 1. \end{aligned}$$

### 3. STATE DISCRIMINATION

Detection area is divided between counters. It is supposed that each counter registers one of the possible states of the system, and there exists one-to-one correspondence between the counters and the states.

Deviations from such correspondence are taken as noise.

#### 3.1. Event space bisection

Definition of probability density is constructed through sequence of bisections of space of events (Rokhlin scheme).

Probability distribution on the event space, according to axiomatics of probability theory, is realized through proceeding to limit in the sequence of bisections of the event space. The initial element of the sequence is the reliability partition – the subset of the

event space for which probability of any outside event is equal to zero. It is supposed that such area has finite measure.

Bisection of the event space is accompanied by adjustment of the methods for description of the states of the particles being detected.

Each section of event space has its own set of states localized in that section.

Now we consider common coordinate space as event space. Description of particles in each section is performed here by means of a set of wave functions localized in respective section.

#### 3.2. Quantum field on section

In the graph (1) typical section of length  $2L$  is shown, and expression for the spatial mode with quantum number  $m$  is  $\phi_m(x, t) = \frac{1}{\sqrt{L}} e^{(ip_m x - i\varepsilon(p_m)t)}$  where  $p_m = \frac{\pi m}{L}$  and  $\varepsilon(p) \equiv \varepsilon_p = \sqrt{\mu^2 + p^2}$ . Quantum field following Fermi statistics is convenient because of limited number of particles in each mode.

$$\hat{\psi}(x, t) = \sum_{m=-\infty}^{\infty} \left( u_{p_m}^{(+)} \phi_m(x, t) \hat{a}_m + u_{p_m}^{(-)} \phi_m^*(x, t) \hat{b}_m^+ \right) \quad (2)$$

The spinors

$$u_p^{(+)} = \begin{pmatrix} \frac{\varepsilon + \mu}{\sqrt{2\varepsilon(\varepsilon + \mu)}} \\ \frac{p}{\sqrt{2\varepsilon(\varepsilon + \mu)}} \end{pmatrix}; \quad u_p^{(-)} = \begin{pmatrix} \frac{-p}{\sqrt{2\varepsilon(\varepsilon + \mu)}} \\ \frac{\varepsilon + \mu}{\sqrt{2\varepsilon(\varepsilon + \mu)}} \end{pmatrix},$$

describe separation of states to positive-frequency and negative-frequency ones.

Hamiltonian of the field is

$$\hat{H} = \sum_{m=-\infty}^{\infty} \hbar \varepsilon(p_m) \left( \hat{a}_m^+ \hat{c}_m + \hat{d}_m^+ \hat{b}_m \right)$$

Operators  $\hat{a}_m$ ,  $\hat{a}_m^+$  generate Heisenberg-Weil algebra for particle modes and  $\hat{b}_m$ ,  $\hat{b}_m^+$  for antiparticle ones.

### 3.3. Quantum field on bisected bases

Bisection of the initial section of length  $2L$  into two parts requires construction of additional representation of quantum field in each of the subsections. Wave functions for the left subsection are shown in the graph (1).

Decrease of length leads to re-definition of the sequences of momenta for each of the modes. The new sequence is  $\{q_m = \frac{2\pi m}{L} : m \in Z\}$ .

Representation in left subsection is given by expansion of quantum field

$$\hat{\psi}(x, t) = \sum_{m=-\infty}^{\infty} \left( u_{q_m}^{(+)} \phi_{m, \text{left}}(x, t) \hat{c}_m + u_{q_m}^{(-)} \phi_{m, \text{left}}^*(x, t) \hat{d}_m^+ \right) \quad (3)$$

Heisenberg-Weil algebra for that representation is generated by operators  $\hat{c}_m$ ,  $\hat{c}_m^+$  and  $\hat{d}_m$ ,  $\hat{d}_m^+$ .

$$\hat{H} = \sum_{m=-\infty}^{\infty} \hbar \varepsilon(q_m) \left( \hat{c}_m^+ \hat{c}_m + \hat{d}_m^+ \hat{d}_m \right)$$

Bisection leads to one more representation associated to the right subsection.

$$\begin{aligned} \hat{\psi}(x, t) = & \sum_{m=-\infty}^{\infty} \left( u_{q_m}^{(+)} \phi_{m, \text{right}}(x, t) \hat{f}_m \right. \\ & \left. + u_{q_m}^{(-)} \phi_{m, \text{right}}^*(x, t) \hat{g}_m^+ \right) \end{aligned} \quad (4)$$

Heisenberg-Weil algebra for that representation is generated by third set of operators  $\hat{f}_m$ ,  $\hat{f}_m^+$  and  $\hat{g}_m$ ,  $\hat{g}_m^+$ .

$$\hat{H} = \sum_{m=-\infty}^{\infty} \hbar \varepsilon(q_m) \left( \hat{f}_m^+ \hat{f}_m + \hat{g}_m^+ \hat{g}_m \right)$$

### 3.4. Bogolubov transform

The operators of creation and annihilation have changed along with perception of particles and antiparticles.

Inverse Fourier transform for subsection bases

$$\begin{aligned} \hat{c}_m &= \left\langle u_{p_m}^{(+)} \phi_{m, \text{left}}(x, t) | \hat{\psi}(x, t) \right\rangle; \\ \hat{d}_m &= \left\langle u_{p_m}^{(-)} \phi_{m, \text{left}}(x, t) | \hat{\psi}^+(x, t) \right\rangle; \\ \hat{f}_m &= \left\langle u_{p_m}^{(+)} \phi_{m, \text{right}}(x, t) | \hat{\psi}(x, t) \right\rangle; \\ \hat{g}_m &= \left\langle u_{p_m}^{(-)} \phi_{m, \text{right}}(x, t) | \hat{\psi}^+(x, t) \right\rangle \end{aligned}$$

and direct Fourier transform for section basis (2) produce interdependence between operators of creation and annihilation

$$\begin{aligned} \hat{c}_m &= \sum_{n=-\infty}^{\infty} \left( \alpha_{m,n} \hat{a}_n + \beta_{m,n}^* \hat{b}_n^+ \right); \\ \hat{d}_m &= \sum_{n=-\infty}^{\infty} \left( -\beta_{m,n}^* \hat{a}_n^+ + \alpha_{m,n} \hat{b}_n \right); \\ \hat{f}_m &= \sum_{n=-\infty}^{\infty} \left( \alpha_{m,n} \hat{a}_n + \beta_{m,n}^* \hat{b}_n^+ \right); \\ \hat{g}_m &= \sum_{n=-\infty}^{\infty} \left( -\beta_{m,n}^* \hat{a}_n^+ + \alpha_{m,n} \hat{b}_n \right) \end{aligned} \quad (5)$$

One of good effects of bisection is in almost complete coincidence of the Bogolubov

transform coefficients for the left and right subsections

$$\begin{aligned}\alpha_{m,k} &= \frac{1}{\sqrt{2}}\delta_{2m,k} - \frac{i}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}A_{n,m,k}\delta_{2n+1,k}; \\ \beta_{m,k} &= W_m\delta_{-2m,k} - \frac{i}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}B_{n,m,k}\delta_{2n+1,k}; \\ A_{n,m,k} &= \frac{(\varepsilon(p_k)+\mu)(\varepsilon(q_m)+\mu)+p_kq_m}{2\sqrt{\varepsilon(p_k)\varepsilon(q_m)(\varepsilon(p_k)+\mu)(\varepsilon(q_m)+\mu)}} \frac{e^{i(\varepsilon(q_m)-\varepsilon(p_k))t}}{n-m+1/2}; \\ B_{n,m,k} &= \frac{p_k(\varepsilon(q_m)+\mu)-q_m(\varepsilon(p_k)+\mu)}{2\sqrt{\varepsilon(p_k)\varepsilon(q_m)(\varepsilon(p_k)+\mu)(\varepsilon(q_m)+\mu)}} \frac{e^{-i(\varepsilon(q_m)+\varepsilon(p_k))t}}{n+m+1/2}; \\ W_m &= \frac{q_m e^{-2i\varepsilon(q_m)t}}{\sqrt{2\varepsilon(q_m)}}.\end{aligned}$$

#### 4. FERMI PARTICLE CREATION

Vacuum state  $|0\rangle$  for section basis  $\hat{a}_m|0\rangle = 0$ ,  $\hat{b}_m|0\rangle = 0$  can not be vacuum for subsection basis since  $\hat{c}_m|0\rangle = \sum \beta_{m,n} b_n^+|0\rangle \neq 0$ .

Entanglement of the modes of particles and antiparticles leads to unitary nonequivalence of the representations under consideration. The state being vacuum with respect to all the annihilation operators of the section is not vacuum with respect to the operators of the left subsection. As the result the vacuum average of the particle or antiparticle number of each mode of the subsection is non-zero.

$$\begin{aligned}\langle \hat{n}_k \rangle &= \langle 0 | \hat{c}_k^+ \hat{c}_k | 0 \rangle = \langle 0 | \hat{d}_k^+ \hat{d}_k | 0 \rangle \\ &= \langle 0 | \hat{f}_k^+ \hat{f}_k | 0 \rangle = \langle 0 | \hat{g}_k^+ \hat{g}_k | 0 \rangle \quad (6) \\ &= \sum_{n=-\infty}^{\infty} |\beta_{n,k}|^2\end{aligned}$$

The graph (2) shows the dependence of the filling numbers on the mode number. Specific parameter is here the ratio of the linear dimension of the section to the Compton

wavelength. With decrease of this ratio the graph goes up near origin more abruptly and more quickly comes to saturation value 0.5. Saturation value corresponds to our notion about total stochasticity of the phenomenon.

#### 4.1. Correlation of Noise

Since the vacuum state of initial section is a pure state, stochasticity of mode filling in subsection representations makes evidence of correlation between noises. The most interesting correlation is correlation between filling numbers for the modes of the left and the right subsections.

$$\begin{aligned}D_{c,k;f,m} &= \left\langle 0 \left| \hat{c}_k^+ \hat{c}_k \hat{f}_m^+ \hat{f}_m \right| 0 \right\rangle \\ &\quad - \left\langle 0 \left| \hat{c}_k^+ \hat{c}_k \right| 0 \right\rangle \left\langle 0 \left| \hat{f}_m^+ \hat{f}_m \right| 0 \right\rangle \quad (7) \\ &= \sum_{l=-\infty}^{\infty} \beta_{l,k} \beta_{l,m}^* \sum_{n=-\infty}^{\infty} \alpha_{n,m} \alpha_{n,k}^*\end{aligned}$$

Correlation function is substantially non-zero for coinciding mode numbers only  $D_{c,m;f,n} \approx \delta_{m,n} D_n$  and comes to values corresponding to almost total correlation between the modes of the subsections shown by graph(3).

So, the states of the particles in the left and the right subsections are entangled.

Unitary non-equivalence of representations in subsections results from restriction of wave packets.

#### 4.2. Overlap of bases

Alternative version is in field description by means of sets of states with incomplete localization. Each such state has a respective wave packet with well-defined average values of coordinate and momentum, and each set of states with equal average values gives quantum field representation associated with a given point of phase space, representations are distributed over the whole phase space. As such a set we use a set of oscillator states

$$\begin{aligned}\varphi_\alpha(p) &= \frac{1}{\sqrt{\sigma\sqrt{2\pi}}} \exp\left(-\frac{(p-p_\alpha)^2}{4\sigma^2} - ipx_\alpha\right) \\ \varphi_{\alpha,n}(p) &= \frac{\left(\frac{d}{dp} + \frac{p}{2\sigma} - \sigma x_\alpha + i\frac{p_\alpha}{2\sigma}\right)^n}{\sqrt{n!}} \varphi_\alpha(p)\end{aligned}\quad (8)$$

Here  $\alpha = \sigma x_\alpha + ip_\alpha/2\sigma$ , and overlap of basis functions is given by

$$\begin{aligned}\int \varphi_\beta^*(p) \varphi_\alpha(p) dp \\ = \exp(-|\alpha - \beta|^2/2 + (\alpha^*\beta - \alpha\beta^*)/2).\end{aligned}$$

Wave functions of field modes

$$\begin{aligned}\phi_{\alpha,n}^{(\pm)}(x, t) \\ = \int u_p^{(\pm)} \varphi_{\alpha,n}(p) \exp(ipx \mp i\varepsilon(p)t) dp\end{aligned}$$

give the set of representations of quantum field associated to each set of wave functions

$$\hat{\psi}(x, t) = \sum_{n=-\infty}^{\infty} \phi_{\alpha,n}^{(+)}(x, t) \hat{a}_{\alpha,n} + \phi_{\alpha,n}^{(-)}(x, t) \hat{b}_{\alpha,n}^+ \quad (9)$$

Communication relations are canonical in each representation

$$\{\hat{a}_{\alpha,n} \hat{a}_{\alpha,m}^+\} = \{\hat{b}_{\alpha,n} \hat{b}_{\alpha,m}^+\} = \delta_{n,m}$$

and non-canonical for operators attributed to different points of phase space. For example

$$\{\hat{a}_{\alpha,0} \hat{a}_{\beta,0}^+\} = \exp(-|\alpha - \beta|^2/2 + (\alpha^*\beta - \alpha\beta^*)/2)$$

All representations have common vacuum state  $\hat{a}_m |0\rangle = 0$  and  $\hat{b}_n |0\rangle = 0$  for all  $n$ . Transforms between representations do not entangle the creation and annihilation operators, thus the vacuum state is common for all the representations – it is not needed here to distinguish between the representations, and there is no vacuum noise.

One-particle state  $|0, 1_0\rangle = \hat{a}_{0,0}^+ |0\rangle$  in the origin  $\alpha = 0$  is not one-particle state out of the origin

$$P_{\beta,0}(1) = \langle 0, 1_0 | \hat{a}_{\beta,0}^+ \hat{a}_{\beta,0} | 0, 1_0 \rangle = e^{-|\beta|^2},$$

and for two-particle state  $|0, 1_0, 1_1\rangle = \hat{a}_{0,0}^+ \hat{a}_{0,1}^+ |0\rangle$  probability of registration

$$P_{\beta,0}(1) = (1 + |\beta|^2) e^{-|\beta|^2},$$

depends on distance from the origin as it is shown in graph (4).

Probability of joint registration of two particles is characterized by correlation function

$$\begin{aligned}C(\alpha, \beta) &= \langle 1_1, 1_0 | \hat{a}_{\beta,0}^+ \hat{a}_{\beta,0} \hat{a}_{\alpha,0}^+ \hat{a}_{\alpha,0} | 1_1, 1_0 \rangle \\ &\quad - \langle 1_1, 1_0 | \hat{a}_{\beta,0}^+ \hat{a}_{\beta,0} | 1_1, 1_0 \rangle \\ &\quad \cdot \langle 1_1, 1_0 | \hat{a}_{\alpha,0}^+ \hat{a}_{\alpha,0} | 1_1, 1_0 \rangle\end{aligned}\quad (10)$$

shown in the following graph (5).

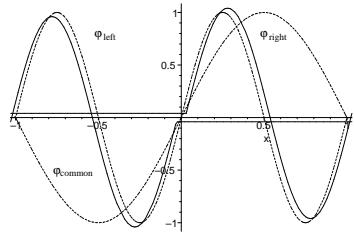
The reason of correlation at registration of particles in given state is in not enough correspondence of detectors simulated by particle number operators to the state. In the case of complete correspondence (the origin) there is no correlation.

## 5. SUMMARY

Imaginarity of entanglement can be caused by one of the next reasons:

- Standard scheme of probability definition leads to the imaginarity of entanglement because of restriction of wave packets;
- Incomplete localization leads to imaginarity of entanglement because of the not enough correspondence between states and detectors.

1. Y. Aharonov, S. Massar, and S. Popescu. Measuring energy, estimating hamiltonians, and the time-energy uncertainty relation. Phys. Rev. A, 66:052107, 2002. quant-ph/0110004.
2. G. Bjork, P. Jonsson, L.L. Sanchez-Soto, Single particle nonlocality and entanglement with vacuum Phys. Rev. A 64, 042106 (2001)
3. M. Popp, F. Verstraete, M. A. Martin-Delgado and J. I. Cirac, Localizable Entanglement- quant-ph/0411123
4. Ph. Blanchard and A. Jadczyk, Relativistic quantum events. Found. Phys., 26:1669, 1996. quant-ph/9610028.
5. Marcin Pawłowski and Marek Czachor, Degree of entanglement as a physically ill-posed problem: The case of entanglement with vacuum - quant-ph/0512253
6. Emanuele Ciancio, Paolo Giorda and Paolo Zanardi, Mode transformations and entanglement relativity in bipartite gaussian states, - quant-ph/0504080
7. Vlatko Vedral, Entanglement in The Second Quantization Formalism - quant-ph/0302040
8. J. B. Hartle. Spacetime coarse grainings in nonrelativistic quantum mechanics. Phys. Rev. D, 44:3173, 1991.
9. V. P. Belavkin, O. Hirota, and R. Hudson, The world of quantum noise and the fundamental output process - quant-ph/0510030

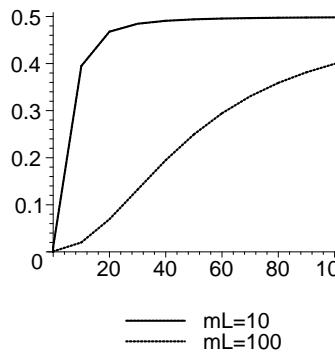


**Figure 1.** Wave functions

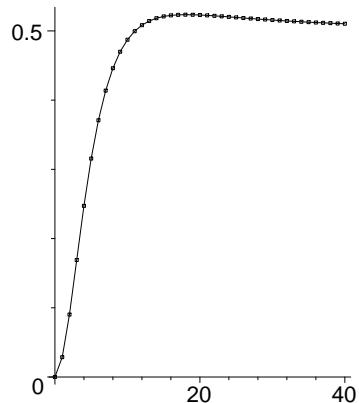
on a section and its subsections.

Wave function of whole section

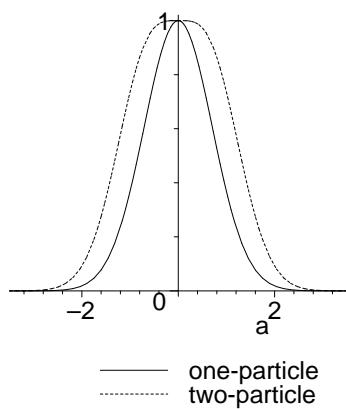
$\phi_{common}$  and the same of sub-sections -  $\phi_{left}$  which vanishes on right subsection and  $\phi_{right}$  - vanishes on left one are shown.



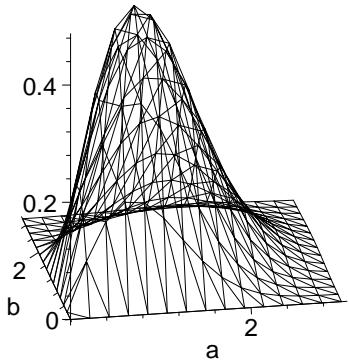
**Figure 2.** Particle creation as bisection result. Average value of particle number versus mode number. Parameter  $mL$  is equal to the ratio of the linear dimension of the section to the Compton wavelength.



**Figure 3.** Correlation function between filling numbers for the modes of the left and the right subsections.



**Figure 4.** Probability of particle registration versus distance from position of state. Two-particle state looks more extensive than the one-particle one.



**Figure 5.** Correlation function of joint registration of two particles for two-particle state. It depends from distances  $a$  and  $b$  between detectors and state origin.